PARTIAL TRANSPOSE OF RANDOM QUANTUM STATES, WEINGARTEN CALCULUS AND MEANDERS

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ABSTRACT. We investigate the asymptotic behavior of the empirical eigenvalues distribution of the partial transpose of a random quantum state. The limiting distribution was previously investigated via Wishart random matrices and shown to be the semicircular distribution or the free difference of two free Poisson distributions, depending on how dimensions of the concerned spaces grow. We use instead Weingarten calculus to study this problem and find three natural regimes in terms of geodesics on the permutation groups. Two of them correspond to the above two cases; the third one turns out to be a new matrix model for the meander polynomials. Moreover, we prove the convergence to the semicircular distribution together with its extreme eigenvalues under weaker assumptions, and show large deviation bound for the latter.

1. INTRODUCTION

In this paper, we investigate asymptotic behavior of the empirical eigenvalues distribution of the *partial transpose* of the random *quantum state* (positive semidefinite Hermitian matrix of trace one) generated by the Haar measure on the unitary group. This problem originated from the field of *quantum information theory* in relation to detecting *entanglement* in a bipartite system. Non-entangled states, called separable states, are necessarily positive semidefinite after partial transpose [Per96], where the latter property is called *Positive Partial Transpose*, abbreviated as *PPT*. The converse statement is not true except for bipartite states on $\mathbb{C}^2 \otimes \mathbb{C}^2$ and $\mathbb{C}^2 \otimes \mathbb{C}^3$ [HHH96]. Here, the partial transpose is made by writing the bipartite matrix as a Kronecker product and transposing each block. Therefore, the generic eigenvalue distribution of the partial transpose of a random quantum state is interesting and especially the behavior of the minimal eigenvalue is important.

Mathematically, we investigate the following problem. Take three complex vector spaces $\mathbb{C}^l$, $\mathbb{C}^m$ and $\mathbb{C}^n$ with $l, m, n \in \mathbb{N}$ and define $\rho = \mathrm{Tr}_{\mathbb{C}^l} |v\rangle\langle v|$

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for a uniformly random unit vector $|v\rangle$ in the product space $\mathbb{C}^l \otimes \mathbb{C}^m \otimes \mathbb{C}^n$. Space $\mathbb{C}^l$ is called the *environment*, space $\mathbb{C}^m \otimes \mathbb{C}^n$ is called the *system*, individual spaces $\mathbb{C}^m$ and $\mathbb{C}^n$ correspond to individual parts of the bipartite system. We will present the details of this construction in Section 2.4. This way of inducing measure on mixed quantum states was investigated in [ŻS01, BŻ06]. Then, we study the asymptotic behavior of the eigenvalues of its partial transpose ρ^Γ where the transpose acts only on the space $\mathbb{C}^n$.

Although quantum states correspond to Hermitian matrices of trace one, Aubrun in [Aub12] used the normal Wishart matrix model to represent a random quantum state. The trace of such a Wishart matrix is a random variable which converges to one only asymptotically. Aubrun showed that the empirical eigenvalues distribution converges to the semicircular distribution as the dimension of the spaces grow in such a way that $l \propto mn$ and $m \propto n$. Later, Banica and Nechita [BN12a, BN12b] showed that the limiting distribution is the free difference of two free Poisson distributions via the Wishart matrix model in the regime where the dimension m of one of the parts of the system is fixed and the dimension l of environment and the dimension n of the other system grow proportionally.

By contrast, we look into this problem more directly by using Weingarten calculus to avoid Wishart matrices and make direct calculations on random quantum states. Then, we calculate the moments to find three natural types of geodesics which yield interesting distributions. Two of them correspond to the above mentioned cases ([Aub12] and [BN12a, BN12b]), and the remaining case turns out to be related to the meander polynomials [DFGG97].

Our paper is organized as follows. After explaining necessary mathematical techniques (Weingarten calculus and free probability) and our precise mathematical model in Section 2, we analyze the regime, where l, m, n grow such that $l \propto mn$ in Section 3. In [Aub12] one requires $m \propto n$ but we drop this condition to get the same limiting measure in Section 3.1, although we need some weak conditions to show the convergence of extreme eigenvalues and their large deviation in Sections 3.2 and 3.4. Then, it is shown in Section 4 that our random matrix model yields the meander polynomials. The connection to free Poisson distribution is presented in Section 5. Section 6 contains the concluding remarks.

2. Preliminaries

2.1. **Free probability, noncrossing partitions and permutations.** In this paper we will use some basic notions from free probability theory. A good treatment of this topic is given in the book [NS06]. We will recall briefly the most important notions.

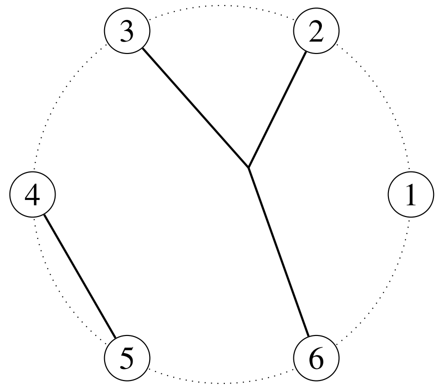


FIGURE 1. A graphical representation of a noncrossing partition $\tau = \big\{\{1\}, \{2,3,6\}, \{4,5\}\big\}$.

2.1.1. *Noncrossing partitions.* A *noncrossing partition* τ is a partition of the set $[n] := \{1, \dots, n\}$ with a property that if $a < b < c < d \in [n]$ are such that a, c belong to the same block of τ and b, d belong to the same block of τ then a, b, c, d belong all to the same block of τ. Noncrossing partitions can be represented graphically as noncrossing connections between points arranged on a circle, see Figure 1.

The set of noncrossing partitions of $[n]$ will be denoted by $\mathrm{NC}(n)$. We also use the notation

$$\mathrm{NC}_{i_1,\dots,i_l}(n) := \big\{\tau \in \mathrm{NC}(n) : |c| \in \{i_1, \dots, i_l\} \quad \forall c \in \tau\big\}$$

for the set of noncrossing partitions of $[n]$ with a restriction on sizes of the blocks.

2.1.2. *Permutations.* We denote the permutation group of n elements by S_n. For a permutation $\alpha \in S_n$ we define $\#\alpha$ to be the number of cycles in α and define its *length* $|\alpha|$ as the minimum number of factors necessary to write α as a product of transpositions. Then the distance in the Cayley graph

$$\mathrm{dist}(\alpha, \beta) := |\alpha^{-1}\beta|$$

is a metric on S_n and

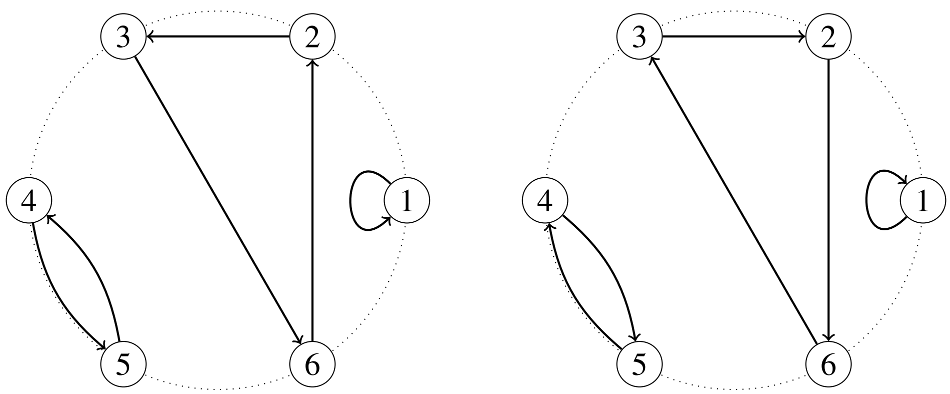
$$\#\alpha = n - |\alpha|$$

holds for all $\alpha \in S_n$.

For given permutations $\alpha, \gamma \in S_n$ we define *the geodesic* $\alpha \to \gamma$ as the set of all permutations which are on the geodesics between α and γ:

$$\alpha \to \gamma := \big\{\beta \in S_n : \mathrm{dist}(\alpha, \beta) + \mathrm{dist}(\beta, \gamma) = \mathrm{dist}(\alpha, \gamma)\big\}.$$

If β belongs to this geodesic, we denote it by $\alpha \to \beta \to \gamma$.



(A) (B)

FIGURE 2. (a) Graphical representation of the permutation $t(\tau) = (1)(236)(45)$ from the geodesic $\mathrm{id} \to \pi = (12\dots6)$ corresponding to the noncrossing partition τ from Figure 1. (b) Graphical representation of the permutation $\big(t(\tau)\big)^{-1} = (1)(632)(54)$ from the geodesic $\mathrm{id} \to \pi^{-1} = (654\dots1)$ corresponding to the noncrossing partition τ from Figure 1.

2.1.3. *Noncrossing partition and permutations.* We will recall now the construction of Biane [Bia97]. We consider the canonical full cycle

$$\pi = (1, 2, \dots, n) \in S_n.$$

For a given partition τ of $[n]$ and $i \in [n]$ we define $\big(t(\tau)\big)(i) \in [n]$ to be the element in the same block of τ as i which is *after* i (with respect to the cyclic order), see Figure 2a. Formally, $\big(t(\tau)\big)(i)$ is the first element of the sequence $\pi(i), \pi^2(i) = \pi\big(\pi(i)\big), \dots$ which belongs to the same block of τ as i. It is easy to check that $t(\tau) \in S_n$.

It is easy to check that $\big(t(\tau)\big)^{-1}(i) \in [n]$ is the element in the same block of τ as i which is *before* i (with respect to the cyclic order), see Figure 2b.

The relationship between noncrossing partitions and geodesics in the Cayley graph of the symmetric groups is given by the following result. Comparison of Figure 1 and Figure 2 is probably the best way to give an intuitive meaning to this relationship.

Lemma 2.1.

a) *The map $\tau \mapsto t(\tau)$ is a bijective correspondence between $\mathrm{NC}(n)$ and the geodesic $\mathrm{id} \to \pi$.*

b) *The map $\tau \mapsto \big(t(\tau)\big)^{-1}$ is a bijective correspondence between $\mathrm{NC}(n)$ and the geodesic $\mathrm{id} \to \pi^{-1}$.*

c) *The map* $\tau \mapsto t(\tau) = \big(t(\tau)\big)^{-1}$ *is a bijective correspondence between* $\mathrm{NC}_{1,2}(n)$ *and the intersection of geodesics* $\big(\mathrm{id} \to \pi\big) \cap \big(\mathrm{id} \to \pi^{-1}\big)$.

Proof. Part *a*) was proved by Biane [Bia97]. Part *b*) follows in a similar way be replacing permutation π by π^{-1}.

We will prove now part *c*). If $\tau \in \mathrm{NC}_{1,2}(n)$ then each cycle of $t(\tau)$ has length 1 or 2, thus $t(\tau) = \big(t(\tau)\big)^{-1}$. From part *a*) and *b*) it follows that $t(\tau) = \big(t(\tau)\big)^{-1}$ belongs to each of the two geodesics.

In order to show surjectivity, let $\sigma \in \big(\mathrm{id} \to \pi\big) \cap \big(\mathrm{id} \to \pi^{-1}\big)$. From part *a*) we know that there exists $\tau \in \mathrm{NC}(n)$ such that $t(\tau) = \sigma$. From part *b*) we know that there exists $\tau' \in \mathrm{NC}(n)$ such that $\big(t(\tau')\big)^{-1} = \sigma$. Since permutations $t(\tau)$ and $t(\tau')$ differ just by orientation of the cycles, it follows that $\tau = \tau'$. Thus $t(\tau) = \sigma = \big(t(\tau)\big)^{-1}$; it follows that σ is an involution, therefore each block of τ consists of 1 or 2 elements. Therefore we showed existence of $\tau \in \mathrm{NC}_{1,2}(n)$ such that $t(\tau) = \sigma$. □

2.1.4. *Genus functions.* It will be convenient to consider the following two non-negative, integer functions on S_n given by:

$$2g_n^{(1)}(\alpha) := \mathrm{dist}(\mathrm{id}, \alpha) + \mathrm{dist}(\alpha, \pi^{-1}) - \mathrm{dist}(\mathrm{id}, \pi^{-1}),$$
$$2g_n^{(2)}(\alpha) := \mathrm{dist}(\mathrm{id}, \alpha) + \mathrm{dist}(\alpha, \pi) - \mathrm{dist}(\mathrm{id}, \pi),$$

for $\alpha \in S_n$. They are called *genus functions*; they measure how the paths via α are longer than the geodesic distance between id and π^{-1} or π.

2.1.5. *Free cumulants.* For a probability measure μ on the real line $\mathbb{R}$ having all moments finite we consider its *free cumulants* $\big(k_p(\mu)\big)_{p=1,2,\dots}$ given by the following implicit relationship with the moments:

$$m_p(\mu) := \int x^p \, d\mu(x) = \sum_{\tau \in \mathrm{NC}(p)} \prod_{b \in \tau} k_{|b|}(\mu). \tag{2.1}$$

For example,

$$\begin{aligned} m_1 &= k_1, \\ m_2 &= k_2 + k_1^2, \\ m_3 &= k_3 + 3k_1k_2 + k_1^3, \\ m_4 &= k_4 + 4k_1k_3 + 2k_2^2 + 6k_1^2k_2 + k_1^4. \end{aligned}$$

Free cumulants are a fundamental tool of the combinatorial approach to free probability theory [NS06].

2.1.6. *Semicircular distribution.* The *semicircular distribution* with mean M and standard variation σ, which will be denoted by $\mathrm{SC}_{M,\sigma}$, has the following density:

$$\frac{d\,\mathrm{SC}_{M,\sigma}}{dx} = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2-(x-M)^2} \qquad \text{for } |x-M| \leq 2\sigma.$$

This measure has a compact support $[M-2\sigma, M+2\sigma]$. The free cumulants of this measure are given by

$$k_p(\mathrm{SC}_{M,\sigma}) = \begin{cases} M & \text{if } p=1, \\ \sigma^2 & \text{if } p=2, \\ 0 & \text{otherwise.} \end{cases}$$

In the special case when $M=1$, the moment-cumulant formula (2.1) implies that the moments of this measure are given by:

$$m_p(\mathrm{SC}_{1,\sigma}) = \sum_{\tau \in \mathrm{NC}_{1,2}(p)} \sigma^{2\ (\#\text{ of blocks of length 2 in } \tau)}.$$

Note that any $\tau \in \mathrm{NC}_{1,2}(p)$ can be identified with an involution $\tau := t(\tau) = \big(t(\tau)\big)^{-1} \in S_p$. By this identification, we have

$$m_p(\mathrm{SC}_{1,\sigma}) = \sum_{\tau \in \mathrm{NC}_{1,2}(p)} \sigma^{2\ |\tau|}. \tag{2.2}$$

2.1.7. *Free Poisson distribution.* Let $\lambda \geq 0$ and $\alpha \in \mathbb{R}$. Then, the *free Poisson distribution* with rate λ and jump-size α is defined to have the following probability density ν on $\mathbb{R}$:

$$\nu_{\lambda,\alpha} = \begin{cases} (1-\lambda)\delta_0 + \lambda\tilde{\nu} & \text{if } 0 \leq \lambda < 1, \\ \tilde{\nu} & \text{if } 1 < \lambda < \infty. \end{cases} \tag{2.3}$$

Here, $\tilde{\nu}_{\lambda,\alpha}$ is the measure supported on the interval $[\alpha(1-\sqrt{\lambda})^2, \alpha(1+\sqrt{\lambda})^2]$ with density:

$$\tilde{\nu}_{\lambda,\alpha}(t) = \frac{1}{2\pi\alpha t}\sqrt{4\lambda\alpha^2-(t-\alpha(1+\lambda))^2}\,dt.$$

Importantly, free cumulants of this distribution are particularly simple: $k_n = \lambda\alpha^n$.

When $\alpha = 1$, the free Poisson distribution is called in particular *Marčenko-Pastur distribution* (with variance 1 and parameter $1/\lambda$).

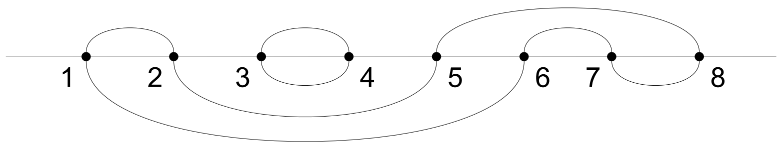


FIGURE 3. Meander of order $n = 4$ with $k = 2$ connected components corresponding to $\sigma_1 = \{\{1,2\},\{3,4\},\{5,8\},\{6,7\}\}$ and $\sigma_2 = \{\{1,6\},\{2,5\},\{3,4\},\{7,8\}\}$.

2.2. **Meanders.** A *meander* of order n is defined as follows. Suppose we have an infinite straight river and $2n$ bridges over the river. Then, a meander is a collection of closed self-avoiding and noncrossing connected roads passing through all of the bridges; in other words, a meander of order n consists of a number of loops crossing a straight line at $2n$ points, see Figure 3. For each $k = 1, 2, \ldots n$, we define $M_n^{(k)}$ to be the number of nonequivalent meanders with k connected components. Another formulation of this object is via noncrossing pair-partitions. If one crosses a bridge then next this person must cross another bridge to come back to the original side of the river; these choices can be represented by an element of $\mathrm{NC}_2(2n)$ for each side of the river. Therefore, equivalent classes of meanders are represented by the elements of $\mathrm{NC}_2(2n) \times \mathrm{NC}_2(2n)$, i.e. pairs of noncrossing pair-partitions.

Then, the meander polynomial $M_n(x)$ is defined to be

$$M_n(x) := \sum_{k=1}^{n} x^k M_n^{(k)}.$$

This polynomial has been extensively investigated and some matrix models were found in [DFGG97]. However, we think that our model can be the simplest matrix model for the meander polynomials (see Theorem 4.2).

2.3. **Weingarten calculus.** A graphical version of Weingarten calculus introduced in [CN10] enables us to transform the algebraic calculus into a graphical one. This method turns out to be very useful, see for example [CFN12]. We briefly review the graphical method in this section.

We have the following formula for integrals of polynomial functions in entries over the unitary group $\mathcal{U}(n)$ with respect to the Haar measure:

Theorem 2.2 (Weingarten [Wei78]). *Let n be a positive integer and $i = (i_1, \ldots, i_p)$, $i' = (i'_1, \ldots, i'_p)$, $j = (j_1, \ldots, j_p)$, $j' = (j'_1, \ldots, j'_p)$ be p-tuples*

of elements of $[n]$. *Then*

$$\int_{\mathcal{U}(n)} U_{i_1 j_1} \cdots U_{i_p j_p} \overline{U_{i'_1 j'_1}} \cdots \overline{U_{i'_p j'_p}} \, dU = \sum_{\alpha,\beta \in S_p} \delta_{i_1 i'_{\alpha(1)}} \cdots \delta_{i_p i'_{\alpha(p)}} \delta_{j_1 j'_{\beta(1)}} \cdots \delta_{j_p j'_{\beta(p)}} \mathrm{Wg}(n, \beta\alpha^{-1}). \tag{2.4}$$

If $p \neq p'$ *then*

$$\int_{\mathcal{U}(n)} U_{i_1 j_1} \cdots U_{i_p j_p} \overline{U_{i'_1 j'_1}} \cdots \overline{U_{i'_{p'} j'_{p'}}} \, dU = 0.$$

In the above theorem, $\mathrm{Wg}(\cdot,\cdot)$ is called *Weingarten function* and it is deeply studied in [CŚ06]. However, in the paper we only use the following property:

Lemma 2.3 (Collins [Col03])**.** *For any value of* n

$$\sum_{\alpha \in S_p} \mathrm{Wg}(n, \alpha) = \prod_{i=0}^{p-1} \frac{1}{n+i}.$$

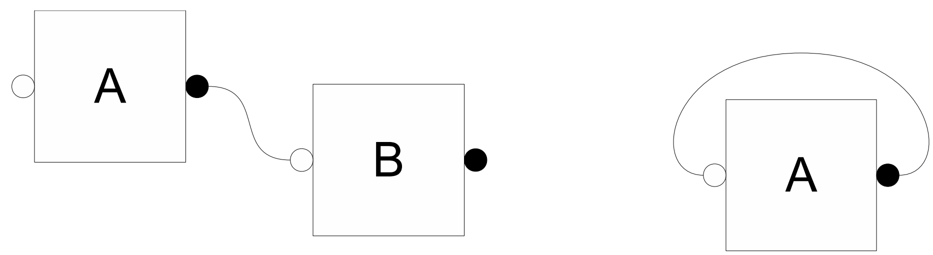
We overview the graphical calculus; we refer the reader to [CN10] for detailed explanations. The graphical calculus consists of boxes and wires. A box represents a vector, a dual vector or a more general tensor (such as a linear map), which is essentially a linear combination of products of vectors and dual vectors. A wire is a generalization multiplication, trace, and similar operations. For example, we view the following two operations as identical in the sense that both are described by a kind of delta function:

$$\begin{aligned} \text{multiplication} \quad &: \quad AB = \sum_{ijkl} a_{ij} b_{kl} \, |i\rangle\langle j|k\rangle\langle l|, \\ \text{trace} \quad &: \quad \operatorname{Tr} A = \sum_{i,j} \langle j|i\rangle \, a_{ij}. \end{aligned}$$

Here, $A = \sum_{ij} a_{ij} \, |i\rangle\langle j|$ and $B = \sum_{kl} \, b_{kl} |k\rangle\langle l|$. In other words, by using wires, we can indicate how the structure of the contractions of the tensors (or, viewed alternatively, indices).

In Figure 4, white circles and black circles represent primal and dual parts of linear maps, respectively. If a linear map acts on a product space, we can use several symbols to represent different spaces. Wires themselves can be interpreted as a generalization of the identity map or the Bell state, etc., but we do not go further in this paper.

The right hand side of (2.4) involves only delta-functions and the Weingarten function. This means that through an integration over the unitary group we can eliminate the boxes of U and $\bar{U}$ and reconnect wires which are



(A) (B)

FIGURE 4. Comparison of (a) multiplication and (b) trace.

attached to those boxes according to the delta-functions on the right hand side of (2.4). This process for fixed permutations α, β is called *removal* and the whole process (which sums over all choices of α, β the corresponding new graphs) is called *graph expansion*.

2.4. **Our model.** Suppose we have three complex vector spaces $\mathbb{C}^l$, $\mathbb{C}^m$ and $\mathbb{C}^n$ with $l, m, n \in \mathbb{N}$ and take the uniformly random unit vector $|v\rangle$ in the product space $\mathbb{C}^l \otimes \mathbb{C}^m \otimes \mathbb{C}^n$. Then, the corresponding random pure state $|v\rangle\langle v|$ on $\mathbb{C}^l \otimes \mathbb{C}^m \otimes \mathbb{C}^n$ induces a random mixed state $\rho = VV^*$ on $\mathbb{C}^m \otimes \mathbb{C}^n$ by tracing out the space $\mathbb{C}^l$. Here, we use the usual isomorphism $|v\rangle \mapsto V$ between $\mathbb{C}^l \otimes \mathbb{C}^m \otimes \mathbb{C}^n$ and $\mathcal{M}_{mn,l}(\mathbb{C})$. We study the asymptotic behavior of the eigenvalues of its partial transpose ρ^Γ with the transpose acting only on the space $\mathbb{C}^n$. More precisely, let $\{\lambda_i\}_{i=1}^{mn}$ be the eigenvalues of the rescaled random matrix $mn\rho^\Gamma$; we define the corresponding empirical eigenvalues distribution

$$\mu_{mn\rho^\Gamma} := \frac{1}{mn} \sum_{i=1}^{mn} \delta_{\lambda_i}(x);$$

our goal is to find the limiting measure in the sense of weak convergence. Note that this scaling is used in Section 3 and some other scalings are chosen in the following sections.

First, we transform ρ into the graphical representation. The uniform random unit vector in $\mathbb{C}^l \otimes \mathbb{C}^m \otimes \mathbb{C}^n$ can be realized concretely as $U\,|e_l\rangle|e_m\rangle|e_n\rangle$ where U is chosen randomly from the unitary group $U(lmn)$ with respect to the Haar measure and $|e_l\rangle$, $|e_m\rangle$ and $|e_n\rangle$ are fixed unit vectors in the respective spaces. Hence, algebraically,

$$\rho = \mathrm{Tr}_{\mathbb{C}^l}\left[U(E_l \otimes E_m \otimes E_n)U^*\right]$$

is a random mixed state on $\mathbb{C}^m \otimes \mathbb{C}^n$, which we are interested in. Here, $E_\lambda = |e_\lambda\rangle\langle e_\lambda|$ for $\lambda = l, m, m$ are appropriate rank one projections. This model of random mixed quantum states was investigated in [ŻS01, BŻ06].

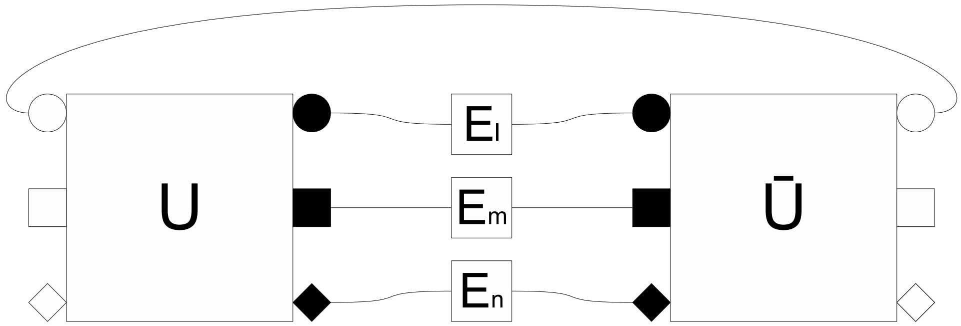


FIGURE 5. A graphical representation of ρ.

The graphical representation of ρ is shown in Figure 5, where circles correspond to $\mathbb{C}^l$, squares to $\mathbb{C}^m$ and diamonds to $\mathbb{C}^n$. For simplicity, we omit those symbols for E_λ matrices ($\lambda = l, m, n$). Importantly, we use $\bar{U}$ instead of U^* since the Weingarten calculus is written in terms of U and $\bar{U}$. As a consequence, we must switch prime and dual parts of the matrix; this explains why the white symbols are outside and the black ones are inside.

2.5. **Moments of ρ^Γ.** For an integer $p \geq 1$ we consider the random variable

$$Z_n^{(p)} := \frac{1}{mn} \operatorname{Tr}\left[\left(mn\rho^\Gamma\right)^p\right] = \int x^p \, d\mu_{mn\rho^\Gamma}$$

which is just the appropriate moment of the empirical eigenvalues distribution $\mu_{mn\rho^\Gamma}$. The following theorem will be our main tool.

Theorem 2.4. *For arbitrary integer $p \geq 1$ the expected value of the corresponding moment of $mn\rho^\Gamma$ is given by*

$$\mathbb{E} Z_n^{(p)} = \mathcal{F}(lmn, p) \sum_{\alpha \in S_p} l^{-|\alpha|} \, m^{p-1-|\pi\alpha|} \, n^{p-1-|\pi^{-1}\alpha|} \tag{2.5}$$

$$= \mathcal{F}(lmn, p) \sum_{\alpha \in S_p} \left(\frac{mn}{l}\right)^{|\alpha|} \, m^{-2g_p^{(1)}} \, n^{-2g_p^{(2)}}, \tag{2.6}$$

where $g_p^{(i)} := g_p^{(i)}(\alpha)$ were defined in Section 2.1.4 and

$$\mathcal{F}(D, p) := D^p \sum_{\gamma \in S_p} \operatorname{Wg}(D, \gamma) = \prod_{i=0}^{p-1} \frac{D}{D+i} = 1 + O\left(D^{-1}\right). \tag{2.7}$$

Proof. Before calculating $\mathbb{E}\operatorname{Tr}(\rho^\Gamma)^p$ we must understand $(\rho^\Gamma)^p$, a part of which is drawn in Figure 6. Notice that the transpose on the space $\mathbb{C}^n$ implies that the connections between white diamonds have a different structure than the connections between white squares.

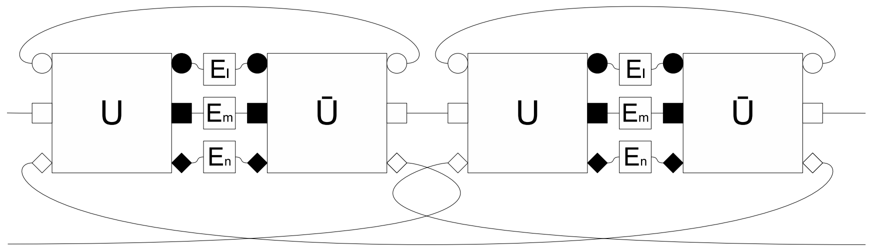


FIGURE 6. A part of a graphical representation of $(\rho^\Gamma)^p$.

Fixed permutations $\alpha, \beta \in S_p$ (as in Theorem 2.2) reconnect white and black symbols respectively via the removal. As a result, the white symbols give the following factors:

- white circles: $l^{\#(\alpha)}$,
- white squares: $m^{\#(\pi\alpha)}$,
- white diamonds: $n^{\#(\pi^{-1}\alpha)}$.

On the other hand, black symbols always give contribution equal to 1. Indeed, since E_λ (for $\lambda = l, m, n$) is a rank-one projection, any resulting loops with boxes, called *necklaces* in [CFN12], always give contribution equal to 1.

Hence,

$$\begin{aligned} \mathop{\mathbb{E}}_{U \in \mathcal{U}(lmn)} \operatorname{Tr}\left[\left(\rho^\Gamma\right)^p\right] &= \sum_{\alpha,\beta \in S_p} l^{\#(\alpha)}\, m^{\#(\pi\alpha)}\, n^{\#(\pi^{-1}\alpha)}\, \mathrm{Wg}(lmn, \alpha^{-1}\beta) \\ &= \mathcal{F}(lmn, p) \sum_{\alpha \in S_p} l^{-|\alpha|}\, m^{-|\pi\alpha|}\, n^{-|\pi^{-1}\alpha|}. \end{aligned}$$

and the result follows by simple algebraic manipulations. □

Interestingly, the above formula (2.5) gives three regimes which give interesting limiting measures. The first case, which we investigate in Section 3, is that $l \propto mn$, where three permutations $\alpha, \pi\alpha, \pi^{-1}\alpha$ interact. The following sections treat the other two cases when l or m are fixed so that just two of the permutations $\alpha, \pi\alpha, \pi^{-1}\alpha$ interact.

3. THE CASE WHEN DIMENSIONS OF ENVIRONMENT AND BOTH SYSTEM PARTS ARE LARGE

In this section we investigate the regime where $l, m, n \to \infty$. Aubrun [Aub12] investigated the case where $l \propto d^2$ and $m, n \propto d$ as $d \to \infty$ via Wishart random matrices model and showed the results which correspond

to Theorem 3.1 and Theorem 3.2. However, we instead investigate asymptotic behavior of the eigenvalues of ρ^Γ in a more general setting by Weingarten calculus. We will show the following results. Theorem 3.1 shows that $mn\rho^\Gamma$ has the limiting distribution as long as the dimension mn of the quantum system $\mathbb{C}^m \otimes \mathbb{C}^n$ is proportional to the dimension l of the environment $\mathbb{C}^l$. Then, Theorem 3.2 studies the behavior of the smallest and largest eigenvalues. They turn out to converge to the two corresponding edges of the support of the limiting density unless m and n grow too differently. Finally, large deviation property of the extreme eigenvalues is investigated in Theorem 3.4 under an additional assumption that m and n are not too close to each other.

3.1. **Limiting eigenvalues.** We analyze the limit of the empirical eigenvalues distribution of $mn\rho^\Gamma$ when $l \propto mn$ where $l, m, n \to \infty$. In the following, we assume without loss of generality that $m \geq n$.

Theorem 3.1. *Suppose* $\frac{mn}{l} \to a$ *with* $0 \leq a < \infty$ *and* $m \geq n$*. Then, as* $n \to \infty$*, the empirical measure of* $mn\rho^\Gamma$ *converges weakly almost surely to the semicircle distribution* $\mathrm{SC}_{1,\sqrt{a}}$*. Here, we think of* $m = m_n$ *and* $l = l_n$ *as sequences which implicitly depend on* n*.*

Before presenting the proof we remark that in the case $a = 0$ the limit distribution $\mathrm{SC}_{1,0} = \delta_1$ becomes a delta measure.

Proof of Theorem 3.1. Non-zero contribution to (2.6) in the limit is given only by the summands for which α is on the following two geodesics:

$$\mathrm{id} \to \alpha \to \pi^{-1}; \qquad \mathrm{id} \to \alpha \to \pi.$$

We apply Lemma 2.1*c*); it follows that

$$\alpha \in \mathrm{NC}_{1,2}(p).$$

Hence, we have

$$\lim_{n\to\infty} \mathbb{E}\, Z_n^{(p)} = \sum_{\alpha \in \mathrm{NC}_{1,2}(p)} a^{|\alpha|} = m_p(\mathrm{SC}_{1,\sigma}) \tag{3.1}$$

with

$$\sigma^2 = a,$$

where we used (2.2). Thus, we proved the convergence in (expected) moments.

To prove almost sure convergence, we will show later that

$$\sum_{n=1}^{\infty} \mathrm{Var}\, Z_n^{(p)} = \sum_{n=1}^{\infty} \mathbb{E}\left[\left(Z_n^{(p)}\right)^2\right] - \left(\mathbb{E}\, Z_n^{(p)}\right)^2 < \infty \tag{3.2}$$

for each $p \in \mathbb{N}$. This result via standard arguments (involving Markov inequality and Borel-Cantelli lemma) would imply that $Z_n^{(p)}$ converges *almost surely* to the appropriate moment of the semicircle distribution. The latter distribution is uniquely determined by its moments, so convergence of measures in the sense of moments implies the convergence in the weak sense; this would finish the proof.

It remains to show that (3.2) indeed holds true; we shall do it in the following. A more careful analysis of (3.1) shows that every term which converges to zero is in fact at most $O\left(n^{-2}\right)$ thus

$$\mathbb{E}\, Z_n^{(p)} = \sum_{\alpha \in \mathrm{NC}_{1,2}(p)} \left(\frac{mn}{l}\right)^{|\alpha|} + O\left(n^{-2}\right).$$

We consider permutation

$$\pi_2 := (1, 2, \ldots, p)(p+1, p+2, \ldots, 2p) \in S_{2p}.$$

One can easily check that $\mathbb{E}\left(Z_n^{(p)}\right)^2$ is equal to the right-hand side of (2.5) and (2.6) in which the summation over S_p is replaced by summation over S_{2p} and permutation π is replaced by π_2. In an analogous way it follows that

$$\mathbb{E}\left(Z_n^{(p)}\right)^2 = \left(\sum_{\alpha \in \mathrm{NC}_{1,2}(p)} \left(\frac{mn}{l}\right)^{|\alpha|}\right)^2 + O\left(n^{-2}\right)$$

which finishes the proof of (3.2). □

Theorem 3.1 shows that the limiting empirical distribution has the compact support on the interval $[1 - 2\sqrt{a}, 1 + 2\sqrt{a}]$. However, this does not necessarily mean that the minimum and maximum eigenvalues converge to the boundaries of this interval. We analyze the convergence of extreme eigenvalues in the next section.

3.2. **Behavior of extreme eigenvalues.** In this section we analyze the behavior of minimum and maximum eigenvalues. Theorem 3.2 below shows that in the regime of Theorem 3.1 the minimum and the maximum eigenvalues of $mn\rho^\Gamma$ actually converge respectively to $1 - 2\sqrt{a}$ and $1 + 2\sqrt{a}$ under an additional condition on growth of m and n.

3.2.1. *Convergence of extreme eigenvalues.*

Theorem 3.2. *Let assumptions of Theorem 3.1 hold true with an additional condition that* $\log m = o(n^{1/6})$. *Then the extreme eigenvalues of* $mn\rho^\Gamma$ *converge to* $1 \pm 2\sqrt{a}$ *almost surely.*

Proof. The difficult part of the theorem is to show that

$$\limsup_n \|mn\rho^\Gamma - I\|_\infty \le 2\sqrt{a}$$

holds almost surely. We will do it in the following.

Let $\{p_n\}_n$ be a sequence of even numbers such that p_n is the largest even number such that $2p_n^{12} \max\left\{1, \frac{mn}{l}\right\} \le n^2$; we write $p = p_n$ for simplicity. Note that then $\log m = o(p)$.

Hence we have

$$\mathbb{E}\,\|mn\rho^\Gamma - I\|_\infty^p \le \mathbb{E}\,\|mn\rho^\Gamma - I\|_p^p = \\ \mathbb{E}\,\mathrm{Tr}\left[(mn\rho^\Gamma - I)^p\right] \le 2mnp^5\left(2\sqrt{\alpha} + o(1)\right)^p.$$

Here, the last inequality follows from Lemma 3.3 below. Therefore, Markov inequality implies that for any $\epsilon > 0$

$$\sum_n \Pr\left\{\|mn\rho^\Gamma - I\|_\infty \ge 2\sqrt{a} + \epsilon\right\} \le \sum_n 2mnp^5\left(\frac{2\sqrt{a} + o(1)}{2\sqrt{a} + \epsilon}\right)^p < \infty.$$

Thus Borel-Cantelli lemma finishes the proof. $\square$

Lemma 3.3. *If* $m \ge n$ *and* $2p^{12}\max\{1, a\} \le n^2$*, then*

$$\mathbb{E}\,\mathrm{Tr}\left[\left(mn\rho^\Gamma - I\right)^p\right] \le 2mnp^5\left(2\sqrt{a} + \frac{\sqrt{a}p}{mn}\right)^p,$$

where $a := \frac{mn}{l}$.

Before presenting the formal proof let us make an intuitive remark on this phenomenon. Since we now know that $mn\rho^\Gamma$ obeys the shifted semicircle law in the limit, we must more or less have the following:

$$\frac{1}{mn}\,\mathbb{E}\,\mathrm{Tr}\left[\left(mn\rho^\Gamma - I\right)^p\right] \approx (\sqrt{a})^p\,\mathrm{Cat}_{p/2} \approx (2\sqrt{a})^p,$$

where used the fact that Catalan numbers

$$\mathrm{Cat}_k = \frac{1}{k+1}\binom{2k}{k} = \int x^{2k}\, d\,\mathrm{SC}_{0,1}(x)$$

are the moments of the centered semicircular distribution.

Proof of Lemma 3.3. First, we expand

$$\frac{1}{mn}\,\mathbb{E}\,\mathrm{Tr}\left[\left(mn\rho^\Gamma - I\right)^p\right] = \sum_{k=0}^{p}\binom{p}{k}(-1)^{p-k}\underbrace{\frac{1}{mn}\,\mathbb{E}\,\mathrm{Tr}\left[\left(mn\rho^\Gamma\right)^k\right]}_{(\spadesuit)}.$$

Here, (2.6) implies that

$$(\spadesuit) = \mathcal{F}(mnl, k)\sum_{\alpha \in S_k} a^{|\alpha|}\, m^{-2g_k^{(1)}} n^{-2g_k^{(2)}}.$$

Next, we decompose S_k in the following way:

$$S_k = \hat{S}_{k,0} \cup \cdots \cup \hat{S}_{k,k},$$

where

$$\hat{S}_{k,t} = \big\{\alpha \in S_k \,:\, \#(\text{non-fixed points of } \alpha) = t\big\}.$$

For example, $S_{k,0} = \{\mathrm{id}\}$ and $S_{k,1} = \emptyset$, etc. We also define another notation:

$$\tilde{S}_t = \hat{S}_{t,t} \subset S_t.$$

For a given $\hat{\alpha} \in \hat{S}_{k,t}$ we consider its *support*, i.e. the set of its non-fixed points and the restriction of $\hat{\alpha}$ to the support. This support has t elements; by a relabeling of these elements, the restriction can be identified with a permutation $\tilde{\alpha} \in \tilde{S}_t$. If the relabeling is chosen to be order-preserving, one can easily check (for example, by removing the fixed points one by one) that

$$\begin{gathered} |\hat{\alpha}| = |\tilde{\alpha}|, \\ \#\left(\hat{\alpha}\pi_k\right) = \#\left(\tilde{\alpha}\pi_t\right), \\ \#\left(\hat{\alpha}^{-1}\pi_k\right) = \#\left(\tilde{\alpha}^{-1}\pi_t\right), \end{gathered}$$

thus the genus functions on $\hat{\alpha}$ and $\tilde{\alpha}$ are the same:

$$g_k^{(i)}(\hat{\alpha}) = g_t^{(i)}(\tilde{\alpha})$$

for $i = 1, 2$.

Hence,

$$\begin{aligned} &\frac{1}{mn}\,\mathbb{E}\left[\left(mn\rho^\Gamma - I\right)^p\right] \\ &= \sum_{k=0}^{p} \binom{p}{k} (-1)^{p-k} \mathcal{F}(mnl,k) \sum_{\alpha \in S_k} a^{|\alpha|} m^{-2g_k^{(1)}(\alpha)} n^{-2g_k^{(2)}(\alpha)} \\ &= \sum_{k=0}^{p} \binom{p}{k} (-1)^{p-k} \mathcal{F}(mnl,k) \sum_{t=0}^{k} \binom{k}{t} \sum_{\alpha \in \tilde{S}_t} a^{|\alpha|} m^{-2g_t^{(1)}(\alpha)} n^{-2g_t^{(2)}(\alpha)} \\ &= \sum_{t=0}^{p} \sum_{\alpha \in \tilde{S}_t} a^{|\alpha|} m^{-2g_t^{(1)}(\alpha)} n^{-2g_t^{(2)}(\alpha)} \sum_{k=t}^{p} \binom{p}{k}\binom{k}{t} (-1)^{p-k} \mathcal{F}(mnl,k) \\ &\qquad\qquad \leq \sum_{t=0}^{p} \binom{p}{t} \left(\frac{\sqrt{a}p}{mn}\right)^{p-t} \underbrace{\sum_{\alpha \in \tilde{S}_t} a^{|\alpha|} m^{-2g_t^{(1)}(\alpha)} n^{-2g_t^{(2)}(\alpha)}}_{(\clubsuit)}. \end{aligned}$$

Here, the last inequality comes from Lemma A.1.

Next, define a function h_t:

$$h_t = g_t^{(1)} + g_t^{(2)}. \tag{3.3}$$

For this function we claim that for $\alpha \in \tilde{S}_t$

$$\frac{t}{2} \leq |\alpha| \leq \frac{t}{2} + h_t(\alpha).$$

Here, the first inequality comes from the definition of $\tilde{S}_t$ and the second one is proved similarly as in (A.4). Then, we have

$$\begin{aligned}(\clubsuit) \leq \sum_{\alpha \in \tilde{S}_t} a^{|\alpha|} n^{-2h_t(\alpha)} &\leq \sum_{h \geq 0} \sum_{\alpha \in T_{t,h}} a^{\frac{t}{2}} \left(\frac{\max\{1,a\}}{n^2} \right)^h \\ &\leq \sum_{h \geq 0} 4^{\frac{t}{2}}\, t^{12h+5}\, a^{\frac{t}{2}} \left(\frac{\max\{1,a\}}{n^2} \right)^h \\ &\leq p^5 \left(2\sqrt{a}\right)^t \sum_{h \geq 0} \left(\frac{p^{12} \max\{1,a\}}{n^2} \right)^h \\ &\leq 2p^5 \left(2\sqrt{a}\right)^t.\end{aligned}$$

Here, the first inequality comes from $m \geq n$, the third one is implied by Lemma A.2.

Thus we showed that

$$\frac{1}{mn} \mathbb{E}\left[\left(mn\rho^\Gamma - I \right)^p \right] \leq \sum_{t=0}^{p} \binom{p}{t} \left(\frac{\sqrt{a}p}{mn} \right)^{p-t} 2p^5 \left(2\sqrt{a}\right)^t.$$

Finally, the binomial formula gives our desired result. □

In the following section, we specify how fast those extreme eigenvalues can converge as is in Theorem 3.4.

3.2.2. *Speed of convergence.* Below, we show a large deviation bound for the extreme eigenvalues by using an extended version of Levy's lemma introduced in [ASW11]. Again, in order to get a nice bound we need further additional condition on the proportion $\frac{mn}{l} \to a$. We prove the result only for the minimum eigenvalue since the proof for the maximum eigenvalue is almost the same.

Theorem 3.4. *In the regime of Theorem 3.2, assume in addition that* $\max\{a, 1/a\} \geq C^2$ *where* C *is as in Lemma B.2. Then, we have the following large deviation bound for the convergence of the minimum eigenvalue of* $mn\rho^\Gamma$ *to* $1 - 2\sqrt{a}$*: there exist three constants* $0 < \epsilon_1, c_1 < 1$ *and* $l_1 \in \mathbb{N}$ *such that*

$$\Pr\left\{ \lambda_{\min} \leq 1 - 2\sqrt{a} - \epsilon \right\} \leq \exp\{-c_1 \epsilon^2 l\}$$

for all $0 < \epsilon < \epsilon_1$ *and* $l \geq l_1$.

Proof. We denote the unit sphere in $\mathbb{C}^d$ by $\mathcal{S}_d$. Then define a function $\lambda_{\min} : \mathcal{S}_{lmn} \to \mathbb{R}$ by:

$$\lambda_{\min}(V) = \big(\text{the minimum eigenvalue of } mn(VV^*)^\Gamma\big) .$$

Here, again we use the identification $|v\rangle = V$ between the two spaces $\mathbb{C}^l \otimes \mathbb{C}^m \otimes \mathbb{C}^n \simeq \mathcal{M}_{mn,l}(\mathbb{C})$. For our purposes, this function must be modified similarly as in [ASW11]. Let $K \subseteq \mathcal{S}_{lmn}$ be the compact subset of the whole domain of $\lambda_{\min}$ defined by

$$K = \left\{ V \in \mathcal{S}_{lmn} \,:\, \|V\|_\infty \leq \frac{2\max\left\{1, \frac{1}{\sqrt{a}}\right\}}{\sqrt{l}} \right\}.$$

First, we claim that the probability of its complement is exponentially small in l:

$$\Pr\left\{V \in K^C\right\} \leq \exp\big(-c\min\{1,a\}\, l\big). \tag{3.4}$$

Here, the constant c is as in Lemma B.2. To verify this bound, we calculate the quantity $\frac{1}{\sqrt{d_1}} + \frac{C}{\sqrt{d_2}}$ which is involved in Lemma B.2. When $a \geq C^2$ we set $d_1 = l$ and $d_2 = mn$ so that

$$\frac{1}{\sqrt{d_1}} + \frac{C}{\sqrt{d_2}} = \frac{1}{\sqrt{l}} + \frac{C}{\sqrt{mn}} \leq \frac{1}{\sqrt{l}} + \frac{1}{\sqrt{l}}.$$

On the other hand, when $\frac{1}{a} \geq C^2$ we let $d_1 = mn$ and $d_2 = l$ and have

$$\frac{1}{\sqrt{d_1}} + \frac{C}{\sqrt{d_2}} = \frac{1}{\sqrt{al}} + \frac{C}{\sqrt{l}} \leq \frac{1}{\sqrt{al}} + \frac{1}{\sqrt{al}}.$$

Next, assume without loss of generality that $\lambda_{\min}(V) \geq \lambda_{\min}(U)$ and denote the normalized eigenvector for the latter by $|u\rangle$. Then, writing $\rho_V = VV^*$, we have

$$\begin{aligned}
\lambda_{\min}(V) - \lambda_{\min}(U) &\leq \langle u|mn\rho_V^\Gamma|u\rangle - \langle u|mn\rho_U^\Gamma|u\rangle \\
&\leq mn\left\|\rho_V^\Gamma - \rho_U^\Gamma\right\|_\infty \\
&\leq mn\left\|\rho_V^\Gamma - \rho_U^\Gamma\right\|_2 \\
&= mn\|\rho_V - \rho_U\|_2 \\
&\leq mn(\|V\|_\infty + \|U\|_\infty)\|V-U\|_2 \\
&\leq 4\max\{a,\sqrt{a}\}\sqrt{l}\|V-U\|_2.
\end{aligned}$$

Hence, the Lipschitz constant of $\lambda_{\min}$ on K is upper-bounded by $4\max\{a,\sqrt{a}\}\sqrt{l}$.

Finally we extend this restricted function to the whole domain by:

$$\tilde{\lambda}_{\min}(V) = \inf_{W\in K}\left\{\lambda_{\min}(W) + 4\max\{a,\sqrt{a}\}\sqrt{l}\,\|V-W\|_2\right\} \qquad \text{for } V \in K^C.$$

This is a modified function of $\lambda_{\min}$ and different from the original only on the small domain K^C. This implies, via Theorem 3.2, that for each $\epsilon > 0$ there exists some $l_0 \in \mathbb{N}$ such that

$$\left(1 - 2\sqrt{a}\right) - \mathrm{Median}\left[\tilde{\lambda}_{\min}\right] < \frac{\epsilon}{2} \qquad \text{for all } l \geq l_0.$$

On the other hand, by applying Lemma B.1 (Levy's lemma) to $\mathcal{S}_{lmn}$ we have

$$\begin{aligned} \Pr_V\left\{\mathrm{Median}\left[\tilde{\lambda}_{\min}\right] - \tilde{\lambda}_{\min}(V) \geq \frac{\epsilon}{2}\right\} &< \exp\left\{-\frac{c_0(2al^2-2)\epsilon^2}{64\max\{a^2,a\}l}\right\} \\ &= \exp\left\{-\frac{c_0(l-(al)^{-1})\epsilon^2}{32\max\{a,1\}}\right\}. \end{aligned} \tag{3.5}$$

To finish the proof, set for example

$$\epsilon_1 = \sqrt{\frac{32\max\{a,1\}\times c\min\{1,a\}}{c_0}} = \sqrt{\frac{32ca}{c_0}}$$

and choose appropriate constants c_1 and l_1 so that applying the union bound method to (3.4) and (3.5) gives the desired result. □

3.2.3. *Implications for quantum information theory.* As already shown by Aubrun [Aub12], $a = 1/4$ is the critical value. The smaller edge of the support of $\mathrm{SC}_{1,1/2}$ is $1 - 2\sqrt{a}$, which is strictly negative when $a > \frac{1}{4}$. In this case Theorem 3.1 implies that the $\limsup$ of the minimum eigenvalue of the matrix ρ^Γ converges almost surely to a strictly negative number. So, our random quantum states are generically not PPT when $a > \frac{1}{4}$. On the other hand, when $a < \frac{1}{4}$, $1 - 2\sqrt{a}$ is by contrast strictly positive. By Theorem 3.2, we know that the probability of quantum states not being PPT is exponentially small in n^2. In other words, our random quantum states are typically PPT when $a < \frac{1}{4}$.

4. THE CASE WHEN THE ENVIRONMENT SPACE IS FIXED AND ITS CONNECTION TO MEANDERS

In this section, we investigate our model when the dimension l of the environment space $\mathbb{C}^l$ is fixed. Unlike the previous regime, we do not have double-geodesics any more. However, interestingly, this regime gives a simple matrix models for the meander polynomials.

4.1. **Our model.**

In this section we investigate the case where

$$l = l_0, \quad \frac{m}{n} \to c,$$

for fixed $l_0 \in \mathbb{N}, c > 0$, in the limit as $n \to \infty$. As usual, $m = m_n$ depends implicitly on n. Then, we are interested in the following empirical distribution of $lm\rho^\Gamma$:

$$\mu_{lm\rho^\Gamma} := \frac{1}{mn} \sum_{i=1}^{mn} \lambda_i, \tag{4.1}$$

where λ_i are the eigenvalues of $lm\rho^\Gamma$. Here we use a different scaling from the one in Section 3. The moments of (4.1) can be written as

$$\frac{1}{mn} \mathop{\mathbb{E}}_{U \in \mathcal{U}(lmn)} \mathrm{Tr}\left[\left(lm\rho^\Gamma\right)^p\right] = \left(1 + O\left(n^{-2}\right)\right) \sum_{\alpha \in S_p} l^{p-|\alpha|} c^{p-1-|\pi\alpha|} n^{p-2-|\pi\alpha|-|\pi^{-1}\alpha|}.$$

Here, we used (2.5) and as before,

$$\begin{aligned} (\text{the power of } n) &= p - 2 - (|\pi\alpha| + |\pi^{-1}\alpha|) \\ &\leq p - 2 - |\pi^2| = \begin{cases} -1 & \text{if } p \text{ is odd,} \\ 0 & \text{if } p \text{ is even.} \end{cases} \end{aligned}$$

Here, as before $\pi = (1, 2, \ldots, p)$ is the canonical full cycle. This means in particular that all the odd moments vanish. When p is even, the bound is satiated if and only if α is on the following geodesic:

$$\pi^{-1} \to \alpha \to \pi.$$

This implies that, for even $p \in \mathbb{N}$,

$$\lim_{n\to\infty} \frac{1}{mn} \mathop{\mathbb{E}}_{U \in \mathcal{U}(lcn^2)} \left[\left(lm\rho^\Gamma\right)^p\right] = \sum_{\pi^{-1} \to \alpha \to \pi} l^{\#\alpha} c^{\#(\pi\alpha)-1}. \tag{4.2}$$

Unfortunately, the above geodesic is very difficult to understand fully. So, we investigate two restricted cases where $l = 1$ or $c = 1$. The former treats random pure states on the bipartite system, and, interestingly, the latter show a connection to the meander polynomials.

4.2. **Random pure states.** We start with the case $l = 1$ of trivial environment space $\mathbb{C}^l$; this corresponds to ρ being a random pure state.

Theorem 4.1. *Let $l = 1$. Then the empirical eigenvalues distribution* (4.1) *converges almost surely as $n \to \infty$ to*

$$\left(1 - \frac{1}{c}\right) \delta_0 + \frac{1}{c} \mu_{B\sqrt{X_1 X_2}} \tag{4.3}$$

in the weak topology of probability measures. Here, $\mu_{B\sqrt{X_1X_2}}$ is the distribution of $B\sqrt{X_1 X_2}$, where the distribution of random variables X_1 and X_2

is the free Poisson distribution $\nu_{1,c}$ as in (2.3)*, and B takes the value 1 or -1 with probability $1/2$, and they are all (classically) independent.*

Before showing the proof let us make two remarks. First, when $c < 1$ the coefficient of δ_0 from the first summand is negative, but this negativity is canceled by the mass from the second summand. Secondly, for $l = 1$ operator ρ^Γ has at most $\min(m,n)^2$ non-zero eigenvalues (this follows from analysis of Schmidt coefficients of ψ, where $\rho = |\psi\rangle\langle\psi|$, see [Aub12, proof of Proposition 9.1]) which explains the atom at zero at the distribution (4.3).

Proof of Theorem 4.1. For a technical reason, we consider the following rescaled empirical distribution:

$$\frac{c}{mn}\sum_{i=1}^{mn}\lambda_i. \tag{4.4}$$

Let $p = 2q$; we consider permutation $\alpha \in S_p$ which contributes to (4.2) so that permutation $\tau := \pi\alpha$ belongs to the following geodesic:

$$\mathrm{id} \to \tau \to \pi^2 = (1,3,\dots,p-1)(2,4,\dots,p).$$

Then the limit moment of (4.4) is equal to $c \times$ (4.2), which can be calculated as follows:

$$c \times (4.2) = \sum_{\mathrm{id}\to\tau\to\pi^2} c^{\#\tau} = \left(\sum_{\tilde\tau\in\mathrm{NC}(q)} c^{\#\tilde\tau}\right)^2.$$

Moreover,

$$\sum_{\tilde\tau\in\mathrm{NC}(q)} c^{\#\tilde\tau} = \sum_{\tilde\tau\in\mathrm{NC}(q)}\prod_{V\in\tau} c$$

coincides with the appropriate moment of the free Poisson distribution $\nu_{c,1}$ (with rate c and jump size 1). Hence, for the even moments, square root of the product of two classically independent free Poisson random variables gives the right moments. However, since all the odd moments vanish, we must add the factor B to recover our desired moments to get the limit distribution of (4.4). After rescaling back this distribution, the additional atom $(1-\frac{1}{c})\delta_0$ does not change the moments m_p of the measure for $p = 1,2,\dots$ but takes care of the correct value of the moment m_0 (the total mass of the measure). This shows convergence in expected moments.

Almost sure convergence can be proven similarly as in the proof of Theorem 3.1. Since the limit is compactly supported, the converges in moments implies convergence in the weak sense. □

In [Aub12], an unpublished result when $l = c = 1$ was quoted and it reads that the limiting distribution is the product of two independent semicircular distribution $\mathrm{SC}_{0,1}$ (with mean 0 and variance 1). Since if Y gives

(A) (B)

FIGURE 7. (a) graphical representation of a noncrossing partition $\{\{1,3,4\}\{2\}\} \in \mathrm{NC}(4)$, (b) the corresponding noncrossing pair-partition from $\mathrm{NC}_2(8)$.

a semicircle law $\mathrm{SC}_{0,1}$ then Y^2 has the Poisson distribution $\nu_{1,1}$, this is a special case of ours.

4.3. **Meander polynomials with our model.** Next, we consider the case where $c = 1$, where our model gives the meander polynomials.

Theorem 4.2. *If $\frac{m}{n} \to c = 1$ and $q \in \mathbb{N}$ then the $2q$-th moment of $\mu_{lm\rho^\Gamma}$ converges as $n \to \infty$ to the meander polynomial $M_q(l)$.*

Proof. First, since $c = 1$

$$(4.2) = \sum_{\tau_1,\tau_2} l^{\#[\pi^{-1}(\tau_1 \oplus \tau_2)]}. \tag{4.5}$$

Here, $\tau_1 \in \mathrm{NC}(\{1,3,\dots,p-1\})$ and $\tau_2 \in \mathrm{NC}(\{2,4,\dots,p\})$.

Next, we recall the well-known bijection between $\mathrm{NC}(q)$ and $\mathrm{NC}_2(2q)$. We represent a noncrossing partition as in Figure 7a. We add two points i_- and i_+ for both sides of each $i \in [q]$, left and right respectively. Then we connect i_+ and j_- if $\alpha(i) = j$. The example is drawn in Figure 7b, where we also use arrows to show the action of the permutation α. This procedure is commonly known as *fattening*.

Finally, we will calculate $\#[\pi^{-1}(\tau_1 \oplus \tau_2)]$. To understand the loop-structure of $\pi^{-1}(\tau_1 \oplus \tau_2)$ we note that τ_1 and τ_2 act in turn; τ_1 acts on odd numbers and τ_2 even numbers, but π^{-1} switches parities. For this reason, we suggest the following graphical representation, see Figure 8.

(1) We draw two parallel horizontal straight lines with odd-numbered points on the upper line and even-numbered points on the lower line.
(2) Draw the graphical representation for τ_1 above the upper line and the graphical representation for τ_2 below the lower line.
(3) Identify respectively $(2i+1)_-$ and $(2i)_+$, and then $(2i)_-$ and $(2i-1)_+$.

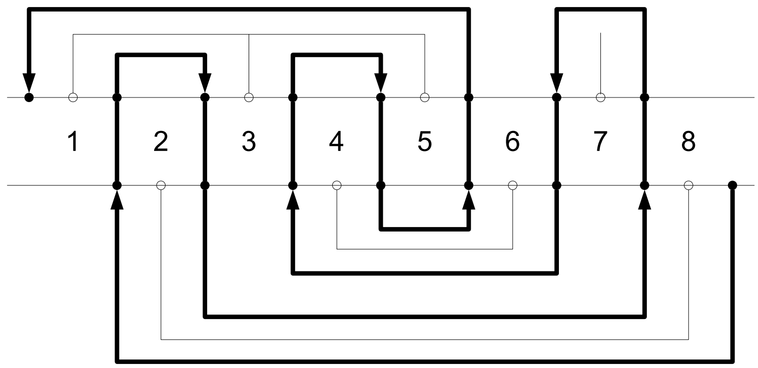


FIGURE 8. Graphical representation of noncrossing partitions $\tau_1 = (1,3,5)(7)$ and $\tau_2 = (2,8)(4,6)$

Here, note that the last step corresponds to the action of π^{-1}. An example with is drawn in Figure 8. Since we can think of τ_1 and τ_2 as elements of $\mathrm{NC}_2(2q)$, identifying 1_{-1} and $2q_+$ reduces (4.5) into the meander polynomials $M_q(l)$. $\square$

5. THE CASE WHEN ONE OF THE SYSTEM PARTS IS FIXED

In this section, we review the case where one of the spaces $\mathbb{C}^m$ or $\mathbb{C}^n$ is fixed. Without loss of generality we may assume that $\mathbb{C}^m$ is fixed. This case was investigated by Banica and Nechita in [BN12a, BN12b] via approximation by the complex Wishart matrix and they proved that the limiting measure is the free difference of two free Poisson distributions. We will present a new proof of this result.

Suppose

$$\frac{l}{n} \to b, \quad m = m_0,$$

where $b > 0$ and $m_0 \in \mathbb{N}$ are fixed constants. Then, the restatement of [BN12a, BN12b] in our setting is:

Theorem 5.1 (Banica, Nechita). *The empirical distribution of $ml\rho^\Gamma$ converges weakly, as $n \to \infty$, almost surely to the probability measure of the free difference of free Poisson distributions with parameters $b(m_0 \pm 1)/2$.*

Proof. First, formula (2.5) gives

$$\frac{1}{mn}\mathop{\mathbb{E}}_{U\in\mathcal{U}(lmn)}\mathrm{Tr}[(ml\rho^\Gamma)^p] = \left(1+O\left(n^{-2}\right)\right)\sum_{\alpha\in S_p} b^p(bn)^{-|\alpha|}m^{p-1-|\pi\alpha|}n^{p-1-|\pi^{-1}\alpha|}.$$

Then, the power of n is bounded as:

$$p-1-(|\alpha|+|\alpha^{-1}\pi|)\leq p-1-|\pi|=0.$$

This implies the following geodesic formula for the limit:

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{mn}\mathop{\mathbb{E}}_{U\in\mathcal{U}(lmn)}\mathrm{Tr}[(ml\rho^\Gamma)^p] &= \sum_{\mathrm{id}\to\alpha\to\pi} b^{\#\alpha}m^{\#(\pi\alpha)-1}\\ &= \sum_{\alpha\in\mathrm{NC}(p)} b^{\#\alpha}m^{e(\alpha)}. \end{aligned} \tag{5.1}$$

Here, for the second equality we used $1+e(\alpha)=\#(\pi\alpha)$, which was proven in [BN12a, Lemma 2.1], where $e(\cdot)$ is the number of blocks whose size is even. The latter formula gives the free cumulants of the limiting measure as

$$k_p=\begin{cases} bm & \text{if } p \text{ is even,}\\ b & \text{if } p \text{ is odd.}\end{cases} \tag{5.2}$$

If we set

$$x=\frac{b(m+1)}{2};\qquad y=\frac{b(m-1)}{2}$$

and define X (resp. Y) to be a random variable with free Poisson distribution $\nu_{x,1}$ (resp. $\mu_{y,1}$) then, if X and Y are free then the cumulants of $X-Y$ are given by

$$k_p^{(X-Y)}=x+(-1)^p y=\begin{cases} bm & \text{if } p \text{ is even,}\\ b & \text{if } p \text{ is odd.}\end{cases}$$

so they coincide with the cumulants (5.2) of the limiting distribution. In this way we showed convergence in expected moments.

Almost sure convergence can be proven similarly as in the proof of Theorem 3.1. As the limiting distribution is compactly supported, the convergence in moments implies weak converge. □

6. CONCLUDING REMARK

In this paper, we investigated asymptotic behavior of eigenvalues of partial transpose of random quantum states on bipartite systems. We naturally picked up three regimes depending on how the concerned spaces grow and showed their connections to the semicircular distribution, the free Poisson

distribution and the meander polynomials. Other regimes may show other interesting behaviors.

APPENDIX A. LEMMAS FOR SECTION 3.2.1

Lemma A.1. *For arbitrary integers $p, t \geq 0$ and $D \geq 2$ the following bound holds true:*

$$\sum_{k \geq 0} \binom{p}{k} \binom{k}{t} (-1)^{p-k} \, \mathcal{F}(D,k) \leq \binom{p}{t} \left(\frac{p}{\sqrt{D}} \right)^{p-t}, \tag{A.1}$$

where $\mathcal{F}$ is given by (2.7).

Proof. We denote by S the *shift operator* on functions of a single variable, i.e. $S : g(\cdot) \mapsto g(\cdot + 1)$ or alternatively $(Sg)(x) = g(x+1)$. Then, writing $\mathcal{F}(k) := \mathcal{F}(D,k)$, the left-hand side of (A.1) becomes

$$\begin{aligned} \sum_{k \geq 0} \binom{p}{k} \binom{k}{t} (-1)^{p-k} \mathcal{F}(k) &= \sum_{k \geq 0} \binom{p}{k} \, S^k \binom{\cdot}{t} \mathcal{F}(\cdot) \, (-1)^{p-k} \bigg|_{\cdot\, =0} \\ &= \Delta^p \binom{\cdot}{t} \mathcal{F}(\cdot) \bigg|_{\cdot\, =0}, \end{aligned} \tag{A.2}$$

where $\Delta = S - 1$ denotes the *forward difference operator*.

Firstly, we will recover the well known product rule for the finite difference Δ. For arbitrary functions g, h of a single variable we have

$$\begin{aligned} \Delta \, [gh] &= (Sg)(Sh) - gh \\ &= [(Sg) - g](Sh) + g \, [(Sh) - h] \\ &= (\Delta g)(Sh) + g(\Delta h) \\ &= M(\Delta g \otimes Sh + g \otimes \Delta h), \end{aligned}$$

where $M(g \otimes h) = gh$ denotes the pointwise multiplication of functions. In other words, we showed that

$$\Delta M = M(\Delta \otimes S + 1 \otimes \Delta).$$

It follows that higher powers of the forward difference operator act on products as follows:

$$\Delta^p(gh) = (\Delta^p M)(g \otimes h) = \sum_{l=0}^{p} \binom{p}{l} \times \Delta^l g \times S^l \Delta^{p-l} h.$$

We used the fact that the operators of shift and the forward difference commute: $\Delta S = S \Delta$.

By applying this general formula in our particular setup we obtain:

$$
\begin{aligned}
\text{(A.2)} &= \Delta^p \binom{\cdot}{t} \mathcal{F}(\cdot)\Big|_{\cdot=0} \\
&= \sum_{l=0}^{p} \binom{p}{l} \times \underbrace{\Delta^l \binom{\cdot}{t}\Big|_{\cdot=0}}_{(\diamondsuit)} \times \left(S^l \Delta^{p-l}\right) \mathcal{F}(\cdot)\Big|_{\cdot=0} \\
&= \binom{p}{t} \left(S^t \Delta^{p-t}\right) \mathcal{F}(\cdot)\Big|_{\cdot=0} \\
&= \binom{p}{t} (\Delta^{p-t}\mathcal{F})(t), \qquad \text{(A.3)}
\end{aligned}
$$

where we used the following property of $(\diamondsuit)$:

$$(\diamondsuit) = \Delta^l \binom{\cdot}{t}\Big|_{\cdot=0} = \binom{0}{t-l} = [l=t].$$

Thus our problem is reduced to estimating the quantity $(\Delta^{p-t}\mathcal{F})(t)$ appearing on the right-hand side of (A.3).

If g is an arbitrary function of a single variable then Δ acts on the product $g \times (S^i\mathcal{F})$ as follows:

$$
\begin{aligned}
\Delta\big[g \times (S^i\mathcal{F})\big](k) &= g(k+1)\mathcal{F}(k+i+1) - g(k)\mathcal{F}(k+i) \\
&= \left[g(k+1) - \left(1 + \frac{k+i}{D}\right) g(k)\right] \mathcal{F}(k+i+1) \\
&= \left[\left(\Delta - \frac{k+i}{D}\right) g\right](k) \times (S^{i+1}\mathcal{F})(k),
\end{aligned}
$$

where it follows from the definition (2.7) that

$$\mathcal{F}(k+i) = \left(1 + \frac{k+i}{D}\right) \mathcal{F}(k+i+1).$$

Hence inductively we get

$$[\Delta^q(g \times \mathcal{F})](k) = \left[\left(\Delta - \frac{k+q-1}{D}\right) \cdots \left(\Delta - \frac{k}{D}\right) g\right] \times (S^q\mathcal{F})(k).$$

We are interested in the special case of this formula for $g = 1$ given by the constant function:

$$[\Delta^q\mathcal{F}](k) = \underbrace{\left[\left(\Delta - \frac{k+q-1}{D}\right) \cdots \left(\Delta - \frac{k}{D}\right) 1\right]}_{(\spadesuit)} \times \mathcal{F}(k+q).$$

We will now analyze the structure of expression (♠). We use the shorthand notation

$$P_i = P_i(k) = -\frac{k+i}{D}.$$

Expression (♠) is a product of q factors, each factor being the sum of two terms. Let us expand this product; each of 2^q resulting summands is a product of the forward difference operators Δ (let us say that there are r factors of this form) and expressions of the form $(P_i)_{0\leq i\leq q-1}$ (let us say that there are $q-r$ factors of this form). Notice that these two expressions do not commute so the order of the factors is important. In the following we will study in detail expressions of this form.

Clearly $\Delta P_i = -\frac{1}{D}$ thus a straightforward application of the product rule shows that

$$\Delta P_{i(1)} \cdots P_{i(\ell)} = \sum_{1\leq r\leq \ell} \Delta\ P_{i(1)} \cdots\ P_{i(r)}\ \cdots P_{i(\ell)+1} =$$

$$- \frac{1}{D} \sum_{1\leq r\leq \ell} P_{i(1)} \cdots P_{i(r-1)} P_{i(r+1)+1} \cdots P_{i(\ell)+1}.$$

The right-hand side can be interpreted as follows: we sum over all ways of matching the forward difference operator Δ with one of the factors $P_{i(r)}$ on its right; this factor is removed and every term P_j on the right of $P_{i(r)}$ should be replaced by P_{j+1}. This matching has been illustrated graphically as an arrow connecting the difference operator Δ with the factor $P_{i(r)}$ on which it acts.

This observation can be extended to general products which we consider. Namely, we sum over all ways of matching r difference operators Δ with factors (P_i) in such a way that each operator Δ is matched with some factor P_i which is on its right and each factor P_i is used at most once. The factors (P_i) which are matched should be removed, each unmatched factor P_j (there are $q-2r$ of them) should be replaced by $P_{j+\delta}$ where δ denotes the number of factors (P_i) which are matched and are on the left from P_j, thus $j+\delta \leq q-1$. Also, there is additional factor $\left(-\frac{1}{D}\right)^r$.

We can illustrate this by an example for $r=2$: one of the summands contributing to the product $P_2\ \Delta\ P_5\ \Delta\ P_8\ P_{13}\ P_{17}\ P_{21}\ P_{25}$ is given by:

$$P_2\ \Delta\ P_5\ \Delta\ P_8\ P_{13}\ P_{17}\ P_{21}\ P_{25} =$$

$$\cdots + P_2\ \Delta\ P_5\ \Delta\ P_8\ P_{13}\ P_{17}\ P_{21}\ P_{25} + \cdots =$$

$$\cdots + \frac{1}{D^2} P_2\ P_5\ P_8\ P_{18}\ P_{27} + \cdots$$

The above two-step procedure (selecting one of 2^q summands with r factors Δ, then summing over the matching $\mathcal{P}$) can be clearly replaced by summing simply over $\mathcal{P}$ which should be a partition of the set $[q]$ with r blocks of length 2 and $q-2r$ blocks of length 1 (the places where the difference operator Δ occur are exactly the left elements of two-element blocks of $\mathcal{P}$). The number of such partitions $\mathcal{P}$ is equal to

$$\frac{q(q-1)\cdots(q-2r+1)}{2^r r!} \leq \frac{q^{2r}}{2^r r!}.$$

It follows that for $k \geq 0$

$$\big|(\spadesuit)\big| \leq \sum_{0\leq r\leq \lfloor \frac{q}{2}\rfloor} \frac{1}{D^r}\frac{q^{2r}}{2^r r!}\left(\frac{k+q}{D}\right)^{q-2r}.$$

Thus for $q = p - t$

$$\text{(A.3)} = \binom{p}{t}(\Delta^q\mathcal{F})(t) = \binom{p}{t}\,(\spadesuit)\;\mathcal{F}(p)$$

and

$$\big|\text{(A.3)}\big| \leq \binom{p}{t}\sum_{0\leq r\leq \lfloor \frac{q}{2}\rfloor} \frac{1}{D^r}\frac{p^{2r}}{2^r r!}\left(\frac{p}{D}\right)^{q-2r} = \binom{p}{t}\sum_{0\leq r\leq \lfloor \frac{q}{2}\rfloor} \frac{p^q}{D^{q-r}}.$$

The sum on the right-hand side is dominated by its last summand multiplied by 2, therefore

$$\big|\text{(A.3)}\big| \leq 2\binom{p}{t}\frac{p^q}{D^{q/2}}.$$

which finishes the proof. $\square$

Lemma A.2. *Let $h = h_p$ be a function defined on S_p as in* (3.3) *and*

$$T_{p,h} = \{\alpha \in \tilde{S}_p : h(\alpha) = h\}.$$

Then

$$|T_{p,h}| \leq 4^{\frac{p}{2}-1} p^{12h+5}.$$

Proof. We assume for a while that p is an even number. We define

$$k(\alpha) = 2|\alpha| - p.$$

It fulfills the bound

$$2h(\alpha) \geq 2|\alpha| + |\pi^2| - 2p + 2 = 2|\alpha| - p = k(\alpha). \tag{A.4}$$

Let σ be an arbitrary permutation and τ be a transposition. If $|\tau\sigma| < |\sigma|$, we say that $\sigma' := \tau\sigma$ was obtained by a *cut*. This corresponds to splitting one of the cycles of σ into two cycles; we say that *the cycle was cut*. If $|\tau\sigma| > |\sigma|$, we say that $\sigma' := \tau\sigma$ was obtained by a *gluing*. This corresponds to merging two of the cycles of σ into one cycle.

Let $\alpha \in T_{p,h}$ be given. If α has a cycle of length at least 3, we cut it; we iterate the procedure until we obtain permutation α' which consists only of cycles of length 1 and 2. The number of cuts we need is upper-bounded by $k(\alpha)$. To see this,

$$\text{(the number of cuts for } \alpha) \leq \sum_{c\in\alpha}(|c|-2) = p - 2\#(\alpha) = k(\alpha).$$

Here, $|c|$ is the cardinality of a cycle c of the permutation α. Importantly, $h(\alpha') \leq h(\alpha)$.

Next, we glue fixed points of α' pairwise to get α'' which consists of p disjoint transpositions; in the language of partition this is a (possibly crossing) pairing. The number of glues is upper-bounded by $k(\alpha)/2$;

$$\text{(the number of glues)} \leq \frac{1}{2}\sum_{c\in\alpha}(|c|-2) = \frac{1}{2}k(\alpha).$$

Above, we used the fact that α has no fixed points.

Write $g(\alpha) = g_p^{(2)}$, which is the notation used in Theorem A.3. Each operation of gluing can increase the genus at most by 1; furthermore $g \leq h$ so that

$$g(\alpha'') \leq g(\alpha') + \frac{1}{2}k(\alpha) \leq 2h(\alpha).$$

To summarize: we constructed a map $\alpha \mapsto \alpha''$ where $\alpha \in T_{p,h}$, with a property that $g(\alpha'') \leq 2h$ and α'' is an involution without fixed points. Furthermore, α can be obtained from α'' by multiplying by at most $3h$ transpositions; this means that the preimage of any α'' consists of at most $\binom{p}{2}^{3h}$ elements. Then, by using Lemma A.4,

$$|T_{ph}| \leq \binom{p}{2}^{3h} \times \sum_{g=0}^{2h} 4^{\frac{p}{2}-1}p^{3g} \leq 4^{\frac{p}{2}-1}p^{12h+3}$$

which finishes the proof in the case when p is even.

When p is odd number, we select some cycle consisting of more than two elements and remove one of these elements. Then, we can do the same surgeries on $\lfloor \frac{p}{2} \rfloor$ points as before. This time, to recover α from α'' we need additional step so that we increase the power of p in the bound by 2. $\square$

Finally we prove Lemma A.4 by using:

Theorem A.3 (Harer-Zagier formula [HZ86]). *Let $\alpha \in S_{2n}$ and define the genus g of α such that*

$$2g(\alpha) = |\alpha| + |\alpha^{-1}\pi| - 2n + 1,$$

where $\pi = (1, 2, \ldots, 2n-1, 2n)$. *Then, the number of involutions without fixed points with genus* g, *denoted by* $\epsilon_g(n)$, *has the following recursive formula:*

$$(n+1)\epsilon_g(n) = 2(2n-1)\epsilon_g(n-1) + (2n-1)(n-1)(2n-3)\epsilon_{g-1}(n-2).$$

Here, the boundary condition for the recurrence is given by $\epsilon_0(n) = \mathrm{Cat}_n$, *the Catalan numbers.*

Lemma A.4. *We have*

$$\epsilon_g(n) \leq 4^{n-1} n^{3g}$$

for $g \geq 0$.

Proof. First, the bound is true for $g = 0$. This comes from the fact that

$$\epsilon_0(1) = \mathrm{Cat}_1 = 1 \quad \text{and} \quad (n+1)\,\mathrm{Cat}_n = 2(2n-1)\,\mathrm{Cat}_{n-1}\,.$$

Next, we assume that the bound is true for $g-1$. Theorem A.3 implies

$$\begin{aligned} \epsilon_g(n) &\leq 4\epsilon_g(n-1) + 4(n-1)^2\epsilon_{g-1}(n-2) \\ &\leq 4\left[4\epsilon_g(n-2) + 4(n-2)^2\epsilon_{g-1}(n-3)\right] + 4(n-1)^2\epsilon_{g-1}(n-2) \\ &\leq \sum_{j=0}^{n-1} 4^{n-j} j^2 \epsilon_{g-1}(j-1). \end{aligned}$$

By the induction hypothesis,

$$\begin{aligned} \epsilon_g(n) &\leq \sum_{j=0}^{n-1} 4^{n-j} j^2 4^{j-2} (j-1)^{3g-3} \\ &\leq 4^{n-2} \sum_{j=0}^{n-1} j^{3g-1} \leq 4^{n-2} \int_0^n x^{3g-1} dx \leq 4^{n-1} n^{3g}. \end{aligned}$$

□

APPENDIX B. MEASURE CONCENTRATION

In this section we collect some results from asymptotic geometric analysis, which we use in this paper.

The following theorem states that continuous functions on high-dimensional unit spheres have almost constant value except for sets of small measure:

Lemma B.1 (Levy's lemma [Lév51])**.** *Let* $f : \mathbb{S}^k \to \mathbb{R}$ *be a function with Lipschitz constant* L, *then there exists a constant* $0 < c_0 < 1$ *and*

$$\Pr\{x \in \mathbb{S}^k : |f(x) - \mathrm{Median}[f]| \geq \epsilon\} < \exp\left\{\frac{-c_0(k-1)\epsilon^2}{L^2}\right\}$$

It is well-known that singular values of random matrices of Gaussian entries are asymptotically concentrated [MP67] if the prime and dual spaces grow in sufficiently different speeds. This phenomenon is also true even when they are normalized to have the Hilbert-Schmidt norm one. Below, we quote the theorem from [ASW11] which was proven via Levy's lemma:

Lemma B.2 (Aubrun, Szarek, Werner)**.** *There exist two constants* $0 < c < 1 < C$ *such that if we take the uniformly random unit vector* $|v\rangle \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ *with* $d_2 \geq C^2 d_1$*. Then, identify* $|v\rangle = V \in \mathcal{M}_{d_1,d_2}$ *and we have*

$$\Pr\left\{\{\text{ singular values of } V\} \not\subset \left[\frac{1}{\sqrt{d_1}} - \frac{C}{\sqrt{d_2}}, \frac{1}{\sqrt{d_1}} + \frac{C}{\sqrt{d_2}}\right]\right\} \leq \exp\{-cd_1\}.$$

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